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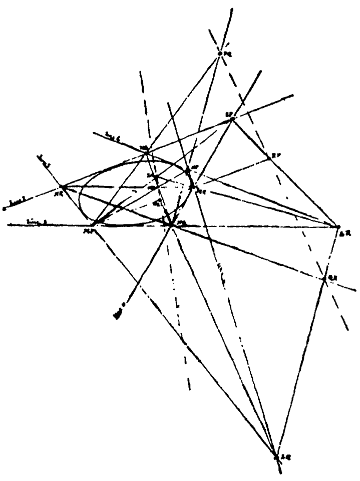
THE DERIVATION OF THE BRIANCHON CONFIGURATION FROM TWO SPATIAL POINT-TRIADS.

By ARCHIBALD HENDERSON, Ph. D., Associate Professor of Mathematics, University of North Carolina, Chapel Hill, N. C.

Cayley* has considered the question of deriving the Pascalian configuration, by projection, from a pair of trihedrals. Denote the three planes of one trihedral by a_1, a_2, a_3; of the other by b_1, b_2, b_3. Considering the nine lines $a_i\,b_j \left\{\begin{matrix} i=1,\ 2,\ 3 \\ j=1,\ 2,\ 3 \end{matrix}\right\}$ and taking them in a particular way in six sets of three each, we may pass hyperboloids through each set of three lines. These hyperboloids intersect in four points O_1, O_2, O_3, O_4 and if we project the solid figure of the two trihedrals from any one of these four points upon an arbitrary plane, the resulting figure is the Pascalian configuration.



This theorem of Cayley's, in connection with certain considerations concerning cubic surfaces, led to the present investigation. Four-point coördinates are used in this paper and a word of explanation is perhaps not amiss. An equation of the form

$$u_1x+u_2y+u_3z+u_4w=0$$

is the equation of a *point*, the coördinates being the variables u_1, u_2, u_3, u_4, which are the perpendiculars from the four points A, B, C, D of the fundamental tetrahedron $ABCD$ upon any plane passing through the point in question. If we have two points given by their equations

$$\xi=0,\ \xi'=0,$$

then the equation

$$\xi+\kappa\xi'=0 \quad (\kappa \text{ constant})$$

represents some point on the line joining the two given points. These preliminary remarks should suffice to make the succeeding discussion readily comprehensible.

Consider two point-triads in space, the one triad consisting of the points designated L, M, N, the other consisting of the points designated P, Q, R. In the system of four-point coördinates chosen above, where u_1, u_2, u_3, u_4 are current coördinates, the equations of the points are chosen as follows:

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*Collected Mathematical Papers, Vol. VI, pp. 129-134.

$$\begin{aligned} L &: u_4=0 \\ M &: mu_1+lu_2+lmnu_3+(mn-l)(nl-m)(lmn-1)u_4=0 \\ N &: nlu_1+mnu_2+u_3-(mn-l)(nl-m)(lmn-1)u_4=0 \end{aligned}$$

and

$$\begin{aligned} P &: u_1-(mn-l)(lmn-1)u_4=0 \\ Q &: u_2-(nl-m)(lmn-1)u_4=0 \\ R &: u_3+(mn-l)(nl-m)u_4=0. \end{aligned}$$

Suppose we are given a point, it is possible to find its polar plane with respect to a surface of the third order. If this surface degenerates into three planes (distinct), the problem is still possible. In like manner, if we are given the equations of three points, in four-point coördinates, it is always possible to determine uniquely the pole of a given plane with respect to this system of three points.

Let v_1, v_2, v_3, v_4 denote constant values of u_1, u_2, u_3, u_4, respectively; $\left.\frac{\partial F}{\partial u_1}\right]_{u_1=v_1} \equiv \frac{\partial F}{\partial v_1}$, and similarly in other cases.

The initial problem is to find a plane such that its pole with respect to the system of points, written in the symbolic form

$$LMN=0\ldots.(1)$$

is identical with its pole with respect to the second system of three points, written

$$PQR=0\ldots.(2)$$

The pole

$$\frac{\partial F}{\partial v_1}u_1+\frac{\partial F}{\partial v_2}u_2+\frac{\partial F}{\partial v_3}u_3+\frac{\partial F}{\partial v_4}u_4=0$$

of the plane (v_1, v_2, v_3, v_4) with respect to the system (1) given by the equation

$$F(u_1, u_2, u_3, u_4)=0$$

has for its equation

$$\begin{aligned} &[2lmnv_1v_4+n(l^2+m^2)v_2v_4+m(l^2n^2+1)v_3v_4+\lambda\mu^2\nu.v_4^2]u_1 \\ &+[2lmnv_2v_4+n(l^2+m^2)v_1v_4+l(m^2+n^2)v_3v_4+\lambda^2\mu\nu.v_4^2]u_2 \\ &+[2lmnv_3v_4+m(l^2n^2+1)v_1v_4+l(m^2+n^2)v_2v_4-\lambda\mu\nu^2.v_4^2]u_3 \\ &+[lmn(v_1^2+v_2^2+v_3^2)-3\lambda^2\mu^2v^2.v_4^2+n(l^2+m^2)v_1v_2+m(l^2m^2+1)v_1v_3 \\ &\qquad+l(m^2+n^2)v_2v_3+2\lambda\mu^2\nu.v_1v_4+2\lambda^2\mu\nu.v_1v_4+2\lambda^2\mu\nu.v_3v_4 \\ &\qquad-2\lambda\mu\nu^2.v_3v_4]u_4=0\ldots.(3) \end{aligned}$$

where λ, μ, $\nu \equiv mn-l$, $nl-m$, $lmn-1$, respectively. Also the pole

$$\frac{\partial \Phi}{\partial v_1}u_1+\frac{\partial \Phi}{\partial v_2}u_2+\frac{\partial \Phi}{\partial v_3}u_3+\frac{\partial \Phi}{\partial v_4}u_4=0$$

of the plane (v_1, v_2, v_3, v_4) with respect to the system (2) given by the equation

$$\Phi(u_1, u_2, u_3, u_4)=0$$

has for its equation

$$\begin{aligned}&[v_2v_3-\mu\nu.v_3v_4+\lambda\mu.v_2v_4-\lambda\mu^2\nu.v_4^2]u_1\\ &+[v_1v_3-\lambda\nu.v_3v_4+\lambda\mu.v_1v_4-\lambda^2\mu\nu.v_4^2]u_2\\ &+[v_1v_2-\lambda\nu.v_2v_4-\mu\nu.v_1v_4+\lambda\mu\nu^2.v_4^2]u_3\\ &+[-\lambda\nu.v_2v_3-\mu\nu.v_1v_3+2\lambda\mu\nu^2.v_3v_4+\lambda\mu.v_1v_2-2\lambda^2\mu\nu.v_2v_4\\ &\qquad -2\lambda\mu^2\nu.v_1v_4+3\lambda^2\mu^2\nu^2.v_4^2]u_4=0\ldots.(4)\end{aligned}$$

where λ, μ, $\nu\equiv mn-l$, $nl-m$, $lmn-1$, respectively, as before.

Now it is evident by inspection that equations (3) and (4) are identical (aside from sign) of

$$v_1=v_2=v_3=0.$$

Accordingly the plane of the face ABC of the fundamental tetrahedron $ABCD$ is such that its pole with respect to the point-triad (1) is coincident with its pole with respect to the point-triad (2).

Connect up the six points L, M, N, P, Q, R by lines and planes in every possible way. Suppose the plane of ABC to be intersected by the line LM in the point LM, and by the plane LMN in the line LMN; and so in the other cases. We obtain in this fashion a configuration in the plane of ABC, consisting of the fifteen $(\equiv {}_2C_6)$ points LM, LN, ...QR, and of the twenty $(\equiv {}_3C_6)$ lines LMN, LMP, ...PQR; and which is such that through each of the points there pass four of the lines, and on each of the lines lie three of the points. Thus the lines

$$\left.\begin{matrix}LMN\\LMP\\LMQ\\LMR\end{matrix}\right\}\text{pass through the point } LM;$$

and the points

$$\left.\begin{matrix}LM\\MN\\NL\end{matrix}\right\}\text{lie on the line } LMN;$$

and so in other cases.

It will next be shown that six lines, denoted 1, 2, 3, 4, 5, 6, may be drawn in the plane of ABC, conditioned as follows:

$$(A)\begin{cases}\text{line (1) passes through the points } LP,\ MQ,\ NR\\ \text{`` (2) `` `` `` } LQ,\ MR,\ NP\\ \text{`` (3) `` `` `` } LR,\ MP,\ NQ\\ \text{`` (4) `` `` `` } LP,\ MR,\ NQ\\ \text{`` (5) `` `` `` } LQ,\ MP,\ NR\\ \text{`` (6) `` `` `` } LR,\ MQ,\ NP\end{cases}$$

For this purpose, represent any line in the plane of ABC as the join of two points whose equations are

$$\left.\begin{array}{l}\lambda_1 u_1+\mu_1 u_2+\nu_1 u_3=0\\ \lambda_2 u_1+\mu_2 u_2+\nu_2 u_3=0\end{array}\right\}$$

If this line meets the line LP, the join of the two points, whose equations are

$$\begin{array}{l}L: u_4=0\\ P: u_1-(mn-l)(lmn-1)u_4=0\end{array}$$

we have the equation of condition

$$\begin{vmatrix}\lambda_1, & \mu_1, & \nu_1, & 0\\ \lambda_2, & \mu_2, & \nu_2, & 0\\ 0, & 0, & 0, & 1\\ 1, & 0, & 0, & -(mn-l)(lmn-1)\end{vmatrix}=0$$

or

$$\mu_1 : \mu_2=\nu_1 : \nu_2,$$

and hence the line in question may be written

$$\left.\begin{array}{c}u_4=0\\ \mu_1 u_2+\nu_1 u_3=0\end{array}\right\}.$$

If this line meets the line MQ also, we have the equation of condition

$$\begin{vmatrix}1, & 0, & 0, & 0\\ 0, & \mu_1, & \nu_1, & 0\\ m, & l, & lmn, & (mn-l)(nl-m)(lmn-1)\\ 0, & 1, & 0, & -(nl-m)(lmn-1)\end{vmatrix}=0$$

or

$$\mu_1 : \nu_1=1 : l$$

and hence the required line has for its equations

$$1:\begin{bmatrix}u_1=0\\ u_2+lu_3=0\end{bmatrix}.$$

If now we write the equation of the point N in the form

$$N : (nlu_1+mnu_2+lmnu_3)-(lmn-1)[u_3+(mn-l)(nl-m)u_4]=0$$

and note the equation of the point *R*

$$R : u_3+(mn-l)(nl-m)u_4=0,$$

it is evident that the equations of the line *NR* may be written

$$NR: \left\{ \begin{array}{l} nlu_1+mnu_2+lmnu_3=0 \\ u_8+(mn-l)(nl-m)u_4=0 \end{array} \right\}$$

That the line 1 meets the line *NR* is now obvious by inspection.

Determining in similar fashion the equations of the five remaining lines, we obtain

$$2: \left\{ \begin{array}{c} u_2=0 \\ mu_1+u_3=0 \end{array} \right\}$$

$$3: \left\{ \begin{array}{c} u_3=0 \\ nu_1+u_2=0 \end{array} \right\}$$

$$4: \left\{ \begin{array}{c} u_1=0 \\ lu_2+u_3=0 \end{array} \right\}$$

$$5: \left\{ \begin{array}{c} u_2=0 \\ u_1+mu_8=0 \end{array} \right\}$$

$$6: \left\{ \begin{array}{c} u_3=0 \\ u_1+nu_2=0 \end{array} \right\}$$

Now these six lines 1, 2, 3, 4, 5, 6 touch the conic given by the equation

$$lmn(u_1^2+u_2^2+u_3^2)+mn(l^2+1)u_2u_3+nl(m^2+1)u_3u_1+lm(n^2+1)u_1u_2=0.$$

This is most easily shown by putting u_1, u_2, u_3 in turn equal to zero; we obtain respectively,

$$mn(u_2+lu_3)(lu_2+u_3)=0,$$
$$nl(mu_1+u_3)(u_1+mu_3)=0,$$
$$lm(nu_1+u_2)(u_1+nu_3)=0.$$

Moreover it is clear from an inspection of the scheme (A) above that the points *LP, LQ, LR*; *MP, MQ, MR*; *NP, NQ, NR* are the points 14, 25, 36; 35, 16, 24; 26, 34, 15, respectively, where 14, for example, denotes the meet of the lines 1 and 4; and so in other cases.

Conversely, starting from the six lines 1, 2, 3, 4, 5, 6 touching the above conic, and denoting the points 14, 25, 36; 35, 16, 24; 26, 34, 15 (which are indeed the vertices, and meets of opposite sides of the six-side 162435) in the manner described above, then it is possible to complete the figure of the fifteen points *LM*, *LN*, ...*QR* and of the twenty lines *LMN*, *LMP*, ...*PQR*, such that through each point pass four lines, and on each line lie three points, as detailed in the foregoing.

Of the fifteen points, nine, viz. the points *LP*, *LQ*, *LR*; *MP*, *MQ*, *MR*; *NP*, *NQ*, *NR* are, as appeared above, points on two of the six lines 1, 2, 3, 4, 5, 6; the remaining points are *MN*, *NL*, *LM*; *QR*, *RP*, *PQ*. These are *Brianchon points*

MN	of the six-side	162435
NL	"	152634
LM	"	142536
QR	"	152436
RP	"	142635
PQ	"	162534,

for the point *MN* is the meet of lines *MNP*, *MNQ*, *MNR*≡*MP*, *NP*; *MQ*, *NQ*; *MR*, *NR*≡35, 26; 16, 34; 24, 15; that is, *MN* is the Brianchon point of the six-side 162435; and similar reasoning verifies the above statements for the rest of the six-lines.

To summarize, we have two sets of three six-sides such that the Brianchon points of each set lie *in linea;* and the two lines so obtained together with the eighteen lines through the six Brianchon points, form a system of twenty lines passing by fours through fifteen points.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

154. Proposed by **F. P. MATZ, Sc. D., Ph. D.,** Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Deduce the Sylvestrian Reciprocant of $ax^3+3bx^2y^2+ay^3+d=0$.

Solution by **B. F. FINKEL, A. M., M. Sc.,** Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Differentiating, dividing by 3, and combining, we have

$$a(x^2+y^2\frac{dy}{dx})+2bxy(y+x\frac{dy}{dx})=0....(1).$$

Repeating the operation, we have